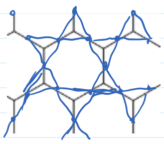
Strategy of the proof:

Consider Partition function:

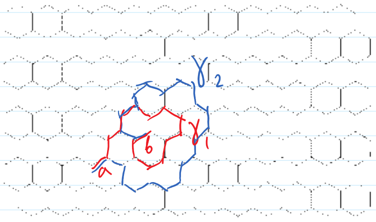
$Z(x) := \sum_{\gamma \text{ starts at } a} x^{-\ell(\gamma)} = \sum_{n \geq 1} c_n x^{-n}$

Enough to prove: $Z(x) < \infty$ iff $x > x_c$ (Root test!)
($x_c = \sqrt{2+\sqrt{2}}$)



All our functions will be defined on edges.
Equivalently, vertices of medial lattice: vertices = edges
Edge - if they share a vertex.

Define rotation of curve γ joining edges a and b as a total rotation of tangent vector between a and b:



$W_{\gamma_1}(a,b) = -2\pi$
$W_{\gamma_2}(a,b) = 2\pi$

Observe: if γ_1 is homotopic to γ_2 inside $\Omega \setminus \{a,b\}$, then $W_{\gamma_1}(a,b) = W_{\gamma_2}(a,b)$.
So if $a, b \in \partial\Omega$, then $W_\gamma(a,b)$ does not depend on γ (Ω - simply connected)!

On hexagonal lattice, on $\gamma = [a = e_0, e_1, \dots, e_n = b]$ can be defined recursively: $W_\gamma[e_0,e_0] = 0$

$$W_\gamma[e_0,e_k] = \begin{cases} W_\gamma[e_0,e_{k-1}] + \frac{\pi}{3}, \text{ if turns left} \\ W_\gamma[e_0,e_{k-1}] - \frac{\pi}{3}, \text{ if turns right} \end{cases}$$

Parafermionic operator ("failed observable")

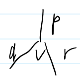
Let $a \in \partial\Omega$, $z \in \Omega$ (edges!).

$$F(z) = F(a, z, x, \sigma) = \sum_{\substack{\gamma \subset \Omega \\ \gamma \text{ - self-avoiding} \\ \gamma \text{ joins } a \text{ to } z}} e^{-i\sigma W_\gamma(a,z)} x^{-\ell(\gamma)}$$

Like a partition function, but weighted by the winding

Key Lemma. $x = x_c = \sqrt{2+\sqrt{2}}$, $\sigma = \frac{5}{8}$

Then for every vertex v inside Ω,



with three adjusent medial vertices p, q, r,

we have $(p-v)F(p) + (q-v)F(q) + (r-v)F(r) = 0$

Proof. Consider all self-avoiding curves from a that end at $p, q,$ or r. Other curves don't contribute to the sum.

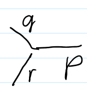
Let $c(\gamma)$ be the contribution of γ to the sum.

r.

Two cases: 1) γ pass through $p, q,$ and r.

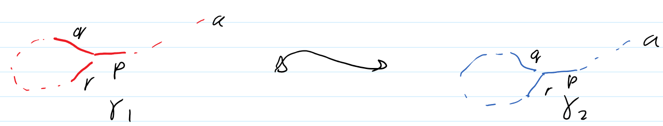
2) All other γ.

Case 1):



$p, q, r \in \gamma \Rightarrow \gamma$ terminates at v!

Pair such curves:



If γ_1 first visits v from p, and exits through q, we just run γ_2 by the same trajectory, but exiting through r.

Computation:

Observe: $\ell(\gamma_1) = \ell(\gamma_2)$

$$\begin{cases} W_{\gamma_1}(a, r) = W_{\gamma_1}(a, p) + W_{\gamma_1}(p, r) = W_{\gamma_1}(a, p) + \frac{4\pi}{3}. \\ W_{\gamma_2}(a, q) = W_{\gamma_2}(a, p) + W_{\gamma_2}(p, q) = W_{\gamma_2}(a, p) - \frac{4\pi}{3}. \end{cases}$$

Does not matter how γ_1 got from a to p:

$\gamma_1(p \to r)$ lies in simply connected domain $\Omega \setminus \gamma(a \to p)$, and both p and r are boundary edges.

So $$c(\gamma_1)+c(\gamma_2) = (r-v)\, e^{-i\sigma W_{\gamma_1}(a,r)}\, x_c^{-\ell(\gamma_1)} + (q-v)\, e^{-i\sigma W_{\gamma_2}(a,q)}\, x_c^{-\ell(\gamma_2)} =$$
$$x_c^{-\ell(\gamma_1)}\, e^{-i\sigma W_{\gamma_1}(a,p)}\, (p-v) \Big(\underbrace{\frac{r-v}{p-v} e^{-i\sigma \frac{4\pi}{3}} + \frac{q-v}{p-v} e^{+i\sigma \frac{4\pi}{3}}}_{\star} \Big) =$$

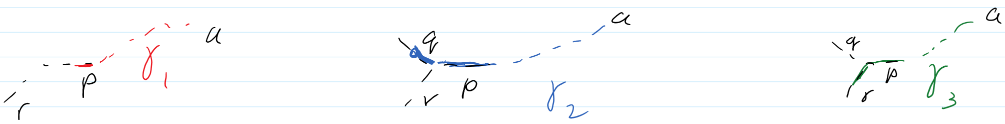
$$\star = e^{-\frac{2\pi i}{3}} e^{-i \frac{5}{6}\pi} + e^{\frac{2\pi i}{3}} e^{i \frac{5}{6}\pi} = e^{-\frac{3\pi}{2} i} + e^{\frac{3\pi}{2} i} = i - i = 0$$

So $\boxed{c(\gamma_1)+c(\gamma_2)=0}$

All of these does not depend on x_c.

This is the way to find right σ.

Case 2): For each curve γ_1 ending at p, there are two curves γ_2 and γ_3, ending at q and r, respectively:



And every pass entering at p can be cut to end at p.

So all case 2) curves separate into triples

Let us observe:

$\ell(\gamma_2) = \ell(\gamma_3) = \ell(\gamma_1)+1$

$W_{\gamma_2}(a,q) = W_{\gamma_2}(a,p) + W_{\gamma_2}(p,q) = W_{\gamma_1}(a,p) - \frac{\pi}{3}$.

$W_{\gamma_3}(a,r) = W_{\gamma_3}(a,p) + W_{\gamma_3}(p,r) = W_{\gamma_1}(a,p) + \frac{\pi}{3}$.

Then

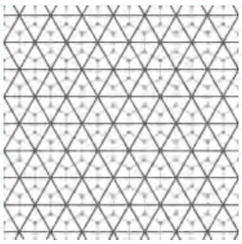
$$c(\gamma_1)+c(\gamma_2)+c(\gamma_3) = (p-v)\, e^{-i\sigma W_{\gamma_1}(a,p)}\, x_c^{-\ell(\gamma_1)}$$

$$\left(1 + x_c^{-1}\,\frac{q-v}{p-v}\, e^{i\frac{5\pi}{24}} + x_c^{-1}\,\frac{r-v}{p-v}\, e^{-i\frac{5\pi}{24}}\right) =$$

$$\ldots \Big(1 + x_c^{-1} \underbrace{\left(e^{\frac{7}{8}\pi i} + e^{\frac{7}{8}\pi i}\right)}_{-2\cos\frac{\pi}{8}}\Big) = 0$$

since $\cos\frac{\pi}{8} = \frac{1}{2x_c} = \frac{1}{2\sqrt{2+\sqrt{2}}}$.

Why is it analiticity?



Consider the dual lattice

vertices: faces

edges = edges.

Dual to hexagonal: triangular.

Discrete integral over dual lattice:

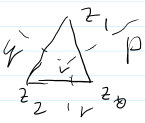
Let $\gamma = [z_0, z_1, \ldots, z_n = z_0]$ be a closed loop on the triangular lattice, f – a function defined on edges.

Then $\oint_\gamma f := \sum_{j=1}^{n} (z_j - z_{j-1})\, f([z_{j-1}, z_j])$.

Restatement of the key Lemma:

For any closed γ, $\oint_\gamma F = 0$. $\quad x = x_c$, $\sigma = \frac{5}{8}$

Proof. It is enough to prove that for every triangle γ, $\oint_\gamma F = 0$.



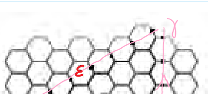
$$\oint = (z_1 - z_0)F(p) + (z_2 - z_1)F(q) + (z_0 - z_2)F(r) =$$

$$\frac{\sqrt{3}}{2} i \left[(p-v)F(p) + (q-v)F(q) + (r-v)F(r) \right] = 0$$

Unfortunately, does not determine F uniquely:

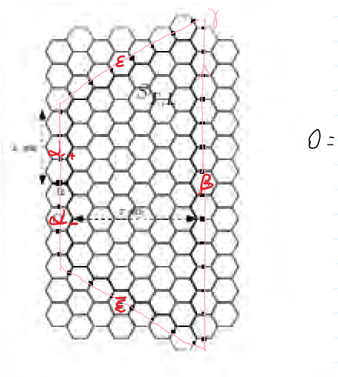
$\frac{2}{3}N$ equations for N edges.

Consider a "lattice trapezoid" $S_{T,L}$: L cells on short side, T cells width



Know: $\oint_\gamma F\,dz = 0$.

1 cells width



Know: $\oint_\gamma F\,dz = 0.$

So we get (after dividing by $-i$):

$$0 = -\sum_{z\in\alpha_+\cup\alpha_-} F(z) + \sum_{z\in\beta} F(z) + e^{i\frac{2\pi}{3}}\sum_{z\in\varepsilon} F(z) + e^{-i\frac{2\pi}{3}}\sum_{z\in\bar\varepsilon} F(z) - F(a)$$

But not that $\arg F(z)$ $(= -\sigma W_\gamma)$ is the same on each part of boundary:

	Phase	σW_γ
β	0	0
α^+	π	$-\frac{5}{8}\pi$
α^-	$-\pi$	$\frac{5}{8}\pi$
ε	$\frac{2\pi}{3}$	$-\frac{5}{12}\pi$
$\bar\varepsilon$	$-\frac{2\pi}{3}$.	$\frac{5}{12}\pi$

Let us now introduce "concatenated partition functions".

$$A^x_{T,L} := \sum_{\substack{\gamma\subset S_{T,L},\ \text{self-avoiding} \\ a\to\alpha_+\cup\alpha_-}} x^{\ell(\gamma)}$$

$$B^x_{T,L} := \text{— // —}\quad a\to\beta$$

$$E^x_{T,L} := \text{— // —}\quad a\to\varepsilon\cup\bar\varepsilon.$$

Let us also observe that by symmetry of $S_{T,L}$, $F(\bar z) = \overline{F(z)}$.

So $$\oint_\alpha F(z) = F(a) + \sum_{z\in\alpha_+\cup\alpha_-} F(z) = F(a) + \frac{1}{2}\sum_{z\in\alpha_+\cup\alpha_-}\left(F(z)+F(\bar z)\right) =$$
$$1 + \frac{e^{-i\frac{\pi}{8}} + e^{i\frac{\pi}{8}}}{2} A^x_{T,L} = 1 - \cos\frac{3\pi}{8} A^x_{T,L}.$$

Same way: $\oint_\beta F\,dz = B^x_{T,L}$

$$\oint_\varepsilon + \oint_{\bar\varepsilon} F\,dz = \frac{e^{i\frac{2\pi}{3}}\cdot e^{i\frac{\pi}{8}\cdot\frac{2\pi}{3}} + e^{-i\frac{2\pi}{3}} e^{-i\frac{\pi}{8}\frac{2\pi}{3}}}{2} E^x_{T_L} = \cos\frac{\pi}{4} E^x_{T_L}.$$

Since we know that for $x = x_c$ the sum of this integrals is zero, we get:

$$\boxed{1 = \cos\frac{3\pi}{8} A^{x_c}_{T,L} + B^{x_c}_{T,L} + \cos\frac{\pi}{4} E^{x_c}_{T,L}}$$

Observe: $A^x_{T,L}$, $B^x_{T,L}$ are increasing in L.

For $x \geq x_c$, $A^x_{T,L} \leq A^{x_c}_{T,L} \leq \dfrac{1}{\cos\frac{3\pi}{8}}$

For $x \geq x_c$, $A^x_{T,L} \leq A^{x_c}_{T,L} \leq \frac{1}{\cos\frac{3\pi}{8}}$

$B^x_{T,L} \leq B^{x_c}_{T,L} \leq 1$.

So, for $x \geq x_c$, $\exists$ $A^x_T := \lim_{L\to\infty} A^x_{T,L}$

$B^x_T := \lim_{L\to\infty} B^x_{T,L}$

By the same relation,

$E^{x_c}_{T,L}$ is decreasing in L, so

$\exists\, E^{x_c}_T := \lim_{L\to\infty} E^{x_c}_{T,L}$

Statement 1. $Z(x_c) = +\infty$ $\left(\Rightarrow \mu \geq \sqrt{2+\sqrt{2}}\right)$

Proof. If for some T, $E^{x_c}_T > 0$, then

$\forall L$ $E^{x_c}_{T,L} \geq E^{x_c}_T$, and

$$Z(x_c) \geq \sum_{L>0} E^{x_c}_{T,L} = \infty.$$

If for all T $E^{x_c}_T = 0$, then by main identity:

$$\cos\frac{3\pi}{8} A^{x_c}_T + B^{x_c}_T = 1.$$

Lemma. $A^{x_c}_{T+1} - A^{x_c}_T \leq \frac{1}{x_c}\left(B^{x_c}_{T+1}\right)^2$

Proof of lemma.

Any walk γ from a to $\alpha_+ \cup \alpha_-$ in S_{T+1} but not in S_T includes a point p adjacent to $\mathcal{B}$ in S_{T+1} (but it does not intersect $\mathcal{B}$!).

Take the first such point p.

Then $\gamma_{a\to p}$ and $\gamma_{p\to\alpha^+\cup\alpha^-}$ are two walks in S_{T+1} almost crossing it. Adding the corresponding vertex of $\mathcal{B}$ creates two walks from $B^{x_c}_{T+1}$ which are one step longer together then γ. Since the decomposition is unique, the lemma follows ∎

Now

$$0 = 1 - 1 = \left(\cos\frac{3\pi}{8} A^{x_c}_{T+1} + B^{x_c}_{T+1}\right) - \left(\cos\frac{3\pi}{8} A^{x_c}_T + B^{x_c}_T\right) \leq$$

$\cos 3\pi$ x_c 2 x x

$$0 = 1-1 = \left(\cos\tfrac{3\pi}{8} A_{T+1}^{x_c} + B_{T+1}^{x_c}\right) - \left(\cos\tfrac{3\pi}{8} A_{T}^{x_c} + B_{T}^{x_c}\right) \le \frac{\cos\frac{3\pi}{8}}{x_c}\left(B_{T+1}^{x_c}\right)^2 + B_{T+1}^{x_c} - B_{T}^{x_c}, \text{ so}$$

$$\frac{\cos\frac{3\pi}{8}}{x_c}\left(B_{T+1}^{x_c}\right)^2 + B_{T+1}^{x_c} \ge B_T^{x_c}.$$

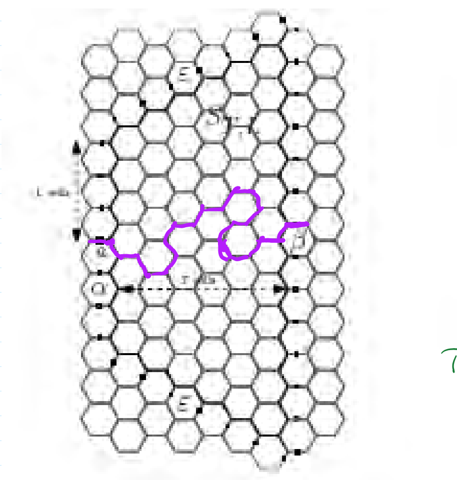
Therefore, by induction on T,

$$B_T^{x_c} \ge T^{-1}\min\left(B_1^{x_c}, \frac{x_c}{\cos\frac{3\pi}{8}}\right), \quad \forall T \ge 1.$$

So $Z(x_c) \ge \sum_{T>0} B_T^{x_c} = \infty$. ∎

Statement 2. $\forall x > x_c : Z(x) < \infty \;(\Rightarrow \mu \ge x_c)$.

Def. Bridge of length T is a horizontal s. a. crossing of S_T, defined up to translation.



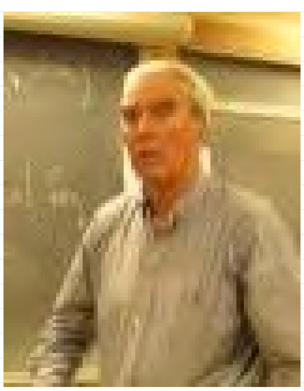
Partition function for bridges is B_T^x.

Since for any T-bridge γ $\ell(\gamma) \ge T$, we have

$$\underbrace{B_T^x}_{\sum_{T\text{-bridges}} x^{-\ell(\gamma)}} \le \left(\frac{x_c}{x}\right)^T B_{T}^{x_c} \le \left(\frac{x_c}{x}\right)^T$$

So $\prod_{T>0}\left(1 + B_T^x\right) < \infty$.

John Hammersly (1920-2004)



Dominic Welsh

Hammersly-Welsh decomposition

SAW can be canonically decomposed into a sequence of bridges of widths $T_{-i} < \dots < T_{-1}$ and $T_0 > \dots > T_j$. And vice versa, for any sequence of such bridges, if one fixes starting midpoint and first edge visited, there is at most one path.

By Hammersly-Welsh:

$$Z(x) = \sum x^{-\ell(\gamma)} \le 2 \sum \left(\prod_{k=-i}^{j} B_{T_k}^x\right) \le \prod\left(1 + B_T^x\right)^2 < \infty.$$

By Hammersly-Welsh:

$$Z(x) = \sum_{\gamma\text{-s.a.}} x^{-\ell(\gamma)} \underset{\text{choice of the first edge}}{\le 2} \sum_{T_{-i}\dots T_0,\dots T_j} \left(\prod_{k=-i}^{j} B^x_{T_k}\right) \underset{a}{=} \prod_{T\ge 0}(1+B^x_T)^2 < \infty.$$

Derivation of Hammersly-Welsh:

First, assume that $\gamma = [a = e_0, \dots, e_n]$, and a its left-most point. γ is in the right half-plane;
Let $[v_0, \dots, v_{n+1}]$ be the vertices of γ. Take $\mathrm{Re}\, v_1 = 0$.
Let $T_0 = \text{width}\ \gamma := \max\{\mathrm{Re}\, v_j\}$.
Let $k = \max\{j : \mathrm{Re}\, v_j = T_0\}$. Then
$\gamma^0 = [a, \dots, e_{k-1}, t]$ is a bridge of size T_0
t added to make it a bridge in S_{T_0}.
Then $[e_{k+1}, \dots, e_n]$ form a walk of width $T_1 < T_0$.
We can use induction.
If γ is reverse half plane walk, i.e.
$\mathrm{Re}\, v_{n+1} = \min \mathrm{Re}\, v_j$, then we can decompose the reverse walk as above with $T_{-1} > T_{-2} > \dots > T_{-i}$.
Finally, for any walk, let $\mathrm{Re}\, v_k = \min\{\mathrm{Re}\, v_j\}$.
Then $[v_0, \dots v_k]$ and $[v_k, \dots v_n]$ are reverse and direct SAW, which can be decomposed.

For uniqueness, once the first vertex is given and first edge is selected, the decomposition determines the walk (just attach them sequentially, and reflect